SYMMETRIZABLE QUANTUM AFFINE SUPERALGEBRAS AND THEIR REPRESENTATIONS

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Abstract

Aspects of the algebraic structure and representation theory of the quantum affine superalgebras with symmetrizable Cartan matrices are studied. The irreducible integrable highest weight representations are classified, and shown to be deformations of their classical counterparts. It is also shown that Jimbo type quantum affine superalgebras can be obtained by deforming universal enveloping algebras of ordinary (i.e., non-graded) affine algebras supplemented by certain parity operators.

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I. Introduction

Quantum affine superalgebras are of great importance for the study of supersymmetric integrable models in statistical mechanics and quantum field theory. Recent research has also indicated that such algebraic structures may play a significant role in characterizing vacua of 4 - dimensional supersymmetric Yang - Mills theories and string compactifications. Apart from their physical applications, quantum affine superalgebras are interesting from a mathematical point of view as well. They have many similarities to the ordinary (i.e., non - graded) quantum affine algebras, thus a thorough investigation of their structures should be possible. It is also hoped that a representation theory can be developed for them, which will be workable in applications. One can quantize the affine superalgebras following Drinfeld and Jimbo relatively easily, once a proper understanding of the Serre type of presentations at the classical level is achieved. However, much more effort seems to be required in order to develop their representation theory, as there already exist severe difficulties at the classical level. Although various special results are known in the area, e.g., the classification of the finite dimensional irreducible representations of $U_q(\widehat{gl}^{(1)}(m|n))$, there has been no attempt to study the quantum affine superalgebras systematically.

The aim of this note is to investigate the structure and representation theory of the quantum affine superalgebras with symmetrizable Cartan matrices. Their classical counterparts, which were classified by Kac[1], constitute the only class of affine superalgebras with a well developed representation theory. One of our results is the classification of the irreducible integrable highest weight representations of these quantum affine superalgebras. We will generalize Lusztig's method [2] to show that such representations are in one to one correspondence with the irreducible integrable highest weight representations of the associated classical affine superalgebras. Another result is that quantum affine superalgebras can be obtained by deforming the universal enveloping algebras of *ordinary* affine algebras supplemented by certain parity operators, wherein some kind of Bose - Fermi transmutation is exhibited. This result will be useful physically, e.g., for showing equivalences of various integrable models. Mathematically, it also bears considerable implications in the classification of quantum affine superalgebras and representation theory. In this note, we will use the result to show a

correspondence between the representations of the super and ordinary quantum affine algebras.

The arrangement of the paper is as follows. In section II we define the Drinfeld type of quantum affine superalgebras and examine some of their algebraic features from the point of view of deformation theory. In section III we classify the integrable highest weight irreps, and in section IV we study the afore mentioned Bose - Fermi transmutation.

II. Quantum Affine Superalgebras

Let $A = (A_{ij})_{i,j=0}^{n}$ be the Cartan matrix of an affine Lie superalgebra which satisfies the following conditions

$$a_{ii} = 2, \qquad a_{ij} \leq 0, \quad i \neq j,$$
$$a_{ij} = 0 \ \text{ iff } \ a_{ji} = 0, \quad a_{ij} \in 2\mathbf{Z}, \quad \text{if} \quad i \in \boldsymbol{\Theta},$$

where $\boldsymbol{\Theta}$ is a nonempty subset of the index set $I = \{0, 1, ..., n\}$. Such Cartan matrices are called symmetrizable, and the affine Lie superalgebras associated with them have been classified. They are given by the following Dynkin diagrams.

Table available upon request

In the above table, a diagram has $n+1$ nodes with the i - th node being white if $i \notin \boldsymbol{\Theta}$, and black if $i \in \boldsymbol{\Theta}$. The i - th and j - th nodes are connected by $\max(|a_{ij}|, |a_{ji}|)$ lines; if $|a_{ij}| > |a_{ji}|$, the lines are endowed with an arrow pointing towards the i - th node. The numerical marks for the diagram will be denoted by a_i, $i = 0, 1, ..., n$, which satisfy the condition $\sum_{j=0}^{n} a_{ij} a_j = 0$.

We denote by $g(A, \boldsymbol{\Theta})$ the complex affine superalgebra associated with the Cartan matrix A and the subset $\boldsymbol{\Theta} \subset \mathbf{I}$. Let H^* be the dual vector space of the Cartan subalgebra of $g(A, \boldsymbol{\Theta})$. Then H^* has a basis $\{\Lambda_0, \alpha_i, \ i \in I\}$, where the α_i are the simple roots. A nondegenerate bilinear form $(.\, , .)$ on H^* can be defined in the standard way, satisfying,

$$\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = a_{ij}, \quad \frac{2(\Lambda_0, \alpha_i)}{(\alpha_i, \alpha_i)} = \delta_{i0}, \quad (\Lambda_0, \Lambda_0) = 0.$$

An appropriate normalization for the form can always be chosen such that

$$(\alpha_\mu,\ \alpha_\mu) \ = \ 1, \quad \forall \mu \in \mathbf{\Theta},$$

and we will work with this normalization throughout.

The Drinfeld model of quantum affine superalgebra $\hat{U}_t(g(A,\mathbf{\Theta}))$ is a $\mathbf{Z}_2$ graded associative algebra over the ring $\mathbf{C}[[t]]$, with $q = exp(t)$, completed with respect to the t - adic topology of $\mathbf{C}[[t]]$. It is generated by the elements $\{d,\ h_i,\ e_i,\ f_i,\ i \in I\}$, subject to the relations

$$\begin{aligned}
k_i k_i^{-1} &= 1, & k_i k_j &= k_j k_i, \\
[d, k_i^{\pm 1}\} &= 0, & & \\
[d, e_i\} &= \delta_{i0} e_i, & [d, f_i\} &= -\delta_{i0} f_i, \\
k_i e_j &= q^{(\alpha_i,\ \alpha_j)} e_j k_i, & k_i f_j &= q^{-(\alpha_i,\ \alpha_j)} f_j k_i, \\
[e_i, f_j\} &= \delta_{ij} \frac{k_i - k_i^{-1}}{q^{\epsilon_i} - q^{-\epsilon_i}}, & & \forall i, j \in I, \\
(Ade_i)^{1-a_{ij}}(e_j) &= 0, & (Adf_i)^{1-a_{ij}}(f_j) &= 0, \quad \forall i \neq j,
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
k_i &= q^{h_i}, \\
\epsilon_i &= \begin{cases} 1, & \text{if } (\alpha_i, \alpha_i) = 1, \\ 1, & \text{if } (\alpha_i, \alpha_i) = 2, \\ 2, & \text{if } (\alpha_i, \alpha_i) = 4. \end{cases}
\end{aligned}$$

All the generators are chosen to be homogeneous, with $d,\ h_i,\ i \in I$, and $e_j,\ f_j,\ j \notin \mathbf{\Theta}$, being even, and $e_\mu,\ f_\mu,\ \mu \in \mathbf{\Theta}$, being odd. For a homogeneous element x, we define $[x] = 0$ if x is even, and $[x] = 1$ when odd. The graded commutator $[.\ ,\ .\}$ represents the usual commutator when any one of the two arguments is even, and the anti commutator when both arguments are odd. The adjoint operation Ad is defined by

$$\begin{aligned}
Ade_i(x) &= e_i x - (-1)^{[e_i][x]} k_i x k_i^{-1} e_i, \\
Adf_i(x) &= f_i x - (-1)^{[f_i][x]} k_i^{-1} x k_i f_i.
\end{aligned}$$

For x being a monomial in e_j's or f_j's, it carries a definite weight $\omega(x) \in H^*$. Then $Ade_i(x) = e_i x - (-1)^{[e_i][x]} q^{(\alpha_i,\ \omega(x))} x e_i$, and similarly for $Adf_i(x)$.

The quantum affine superalgebra $\hat{U}_t(g(A, \mathbf{\Theta}))$ has the structures of a $\mathbf{Z}_2$ graded Hopf algebra with a co - multiplication

$$\begin{aligned}
\Delta(d) &= d \otimes 1 + 1 \otimes d, \\
\Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i, \\
\Delta(e_i) &= e_i \otimes k_i + 1 \otimes e_i, \\
\Delta(f_i) &= f_i \otimes 1 + k_i^{-1} \otimes h_i.
\end{aligned}$$

A co - unit and an antipode also exit, but we shall not spell them out explicitly, as they will not be used here. Our main concern in this letter is the algebraic structures and the representations of the quantum affine superalgebras.

An important fact is that as a $\mathbf{Z}_2$ graded associative algebra, $\hat{U}_t(g(A, \mathbf{\Theta}))$ is a deformation of the universal enveloping algebra $U(g(A, \mathbf{\Theta}))$ of $g(A, \mathbf{\Theta})$ in the sense of [3], that is, being a topologically free $\mathbf{C}[[t]]$ module, $\hat{U}_t(g(A, \mathbf{\Theta}))$ is isomorphic to the $\mathbf{C}[[t]]$ module $U(g(A, \mathbf{\Theta}))[[\mathbf{t}]]$, consisting of power series in t with coefficients in $U(g(A, \mathbf{\Theta}))$, and there also exists the algebra isomorphism $\hat{U}_t(g(A, \mathbf{\Theta}))/\mathbf{t}\hat{\mathbf{U}}_\mathbf{t}(\mathbf{g}(\mathbf{A}, \mathbf{\Theta})) \cong \mathbf{U}(\mathbf{g}(\mathbf{A}, \mathbf{\Theta}))$. This of course is a standard fact in the theory of quantum groups [4] [5]. However, proving it is a highly nontrivial matter, and is well out of the scope of this letter.

To make things more explicit, let m be the associative multiplication of $U(g(A, \mathbf{\Theta}))$. Denote by m_t the associative multiplication of $\hat{U}_t(g(A, \mathbf{\Theta}))$. Then m_t is a $\mathbf{C}[[t]]$ bi - linear map $m_t : U(g(A, \mathbf{\Theta}))[[\mathbf{t}]]\hat{\otimes}\mathbf{U}(\mathbf{g}(\mathbf{A}, \mathbf{\Theta}))[[\mathbf{t}]] \rightarrow U(g(A, \mathbf{\Theta}))[[\mathbf{t}]]$ of the form $m_t = m + \sum_{i=1}^{\infty} t^i m^{(i)}$, where m is the multiplication of $U(g(A, \mathbf{\Theta}))$, and $\hat{\otimes}$ is the tensor product completed with respect to the t - adic topology of $\mathbf{C}[[t]]$. The $m^{(i)} : U(g(A, \mathbf{\Theta})) \otimes U(g(A, \mathbf{\Theta})) \rightarrow \mathbf{U}(\mathbf{g}(\mathbf{A}, \mathbf{\Theta}))$ are $\mathbf{Z}_2$ graded vector space maps, which are homogeneous of degree zero. Associativity of m_t imposes stringent conditions on the maps $m^{(i)}$. In particular, the first nonvanishing $m^{(i)}$ must be a 2 - cocycle in the language of Hochschild cohomology. In view of the fact that the Drinfeld quantum affine algebras are nontrivial deformations of the universal enveloping algebras of the associated affine algebras, we expect the deformations defining the quantum affine superalgebras also to be nontrivial.

Consider a $\hat{U}_t(g(A, \mathbf{\Theta}))$ module V_t, with the module action denoted by $\circ_t$. If V_t is a free $\mathbf{C}[[t]]$ module, then $V_t = V[[t]]$, with $V = V_t/tV_t$ a complex vector space. Assume

that for any given $a \in U(g(A,\mathbf{\Theta})) \subset \hat{\mathbf{U}}_{\mathbf{t}}(\mathbf{g}(\mathbf{A},\mathbf{\Theta}))$, $v \in V \subset V_t$,

$$a \circ_t v \;=\; a \circ v + o(t) \in V[[t]],$$

where $\circ$ represents a $\mathbf{C}$ bi - linear map $U(g(A,\mathbf{\Theta})) \otimes \mathbf{V} \to \mathbf{V}$, then $\circ$ defines a module action of $U(g(A,\mathbf{\Theta}))$ on V. To see that our claim is indeed correct, consider another element $b \in U(g(A,\mathbf{\Theta})) \subset \hat{U}_t(g(A,\mathbf{\Theta}))$. Then

$$\begin{aligned} b \circ_t [a \circ_t (v + tV_t)] &= m_t(b,a) \circ_t v + tV_t \\ &= m(b,a) \circ v + tV_t. \end{aligned}$$

Conversely, let the complex vector space V be a $U(g(A,\mathbf{\Theta}))$ module, with the module action $\circ$. If there exists a $\mathbf{C}[[t]]$ bi - linear map $\circ_t : U(g(A,\mathbf{\Theta}))[[\mathbf{t}]]\hat{\otimes}\mathbf{V}[[\mathbf{t}]] \to \mathbf{V}[[\mathbf{t}]]$, such that for any $a, b \in U(g(A,\mathbf{\Theta})) \subset \hat{\mathbf{U}}_{\mathbf{t}}(\mathbf{g}(\mathbf{A},\mathbf{\Theta}))$, $v \in V \subset V[[t]]$,

$$\begin{aligned} a \circ_t v &= a \circ v + o(t) \in V[[t]], \\ a \circ_t (b \circ_t v) &= m_t(a,b) \circ_t v, \end{aligned}$$

then $V[[t]]$ furnishes a $\hat{U}_t(g(A,\mathbf{\Theta}))$ module. In this case, we say that the $\hat{U}_t(g(A,\mathbf{\Theta}))$ module $(V[[t]], \circ_t)$ is a deformation of the $U(g(A,\mathbf{\Theta}))$ module $(V, \circ)$, and the representation of $\hat{U}_t(g(A,\mathbf{\Theta}))$ afforded by $(V[[t]], \circ_t)$ the deformation of the representation of $U(g(A,\mathbf{\Theta}))$ furnished by $(V, \circ)$. (Note the difference between our definition of deformation of modules and that of [6].) We will call the deformation *trivial* if there exists a $\mathbf{C}[[t]]$ linear map $\Phi_t = id + t\phi_1 + t^2\phi_2 + ... : U(g(A,\mathbf{\Theta}))[[\mathbf{t}]] \to \mathbf{U}(\mathbf{g}(\mathbf{A},\mathbf{\Theta}))[[\mathbf{t}]]$ such that $a \circ_t v = \Phi_t(a) \circ v$, $\forall a \in \hat{U}_t(g(A,\mathbf{\Theta}))$, $v \in V[[t]]$, where $\circ$ is $\mathbf{C}[[t]]$ - linearly extended to $\hat{U}_t(g(A,\mathbf{\Theta}))$. Needless to say, not all representations of $U(g(A,\mathbf{\Theta}))$ can be deformed into representations of $\hat{U}_t(g(A,\mathbf{\Theta}))$. It is a very interesting problem to characterize the deformability of a $U(g(A,\mathbf{\Theta}))$ module in cohomological terms, and we hope to return to the problem in the future. We should also mention that if an irreducible representation can be deformed at all, then the deformation must be trivial.

III. Integrable Highest Weight Modules

Let us first construct the irreducible highest weight $\hat{U}_t(g(A,\mathbf{\Theta}))$ modules. We will omit the symbols m_t and $\circ_t$ from our notations whenever confusion is not likely to

arise. Let U_q^+ be the $\mathbf{Z}_2$ graded subalgebra of $\hat{U}_t(g(A,\mathbf{\Theta}))$ generated by the h_i, e_i, $i = 0, 1, ..., n$, together with d, and N_q^- that generated by the f_i, $i = 0, 1, ..., n$. Let $v_+^\Lambda \otimes \mathbf{C}[[t]]$ be a one dimensional U_q^+ module satisfying

$$h_i v_+^\Lambda = (\Lambda, \alpha_i)\ v_+^\Lambda, \quad e_i v_+^\Lambda = 0, \quad \forall i = 0, 1, ..., n,$$
$$d v_+^\Lambda = 0.$$

We construct the $\mathbf{C}[[t]]$ module

$$\overline{V}_t(\Lambda) \quad = \quad \hat{U}_t(g(A,\mathbf{\Theta})) \otimes_{\mathbf{U_q^+}} \mathbf{v_+^\Lambda},$$

which is clearly isomorphic to $N_q^- \otimes v_+^\Lambda$, and therefore, is spanned by the elements of the form $f_{i_1} f_{i_2} ... f_{i_p} \otimes v_+^\Lambda$, $i_s \in I$, $p \in \mathbf{Z}_+$. (We will ommit the tensor product sign $\otimes$ from such expressions hereafter.) Define a bi - linear action of $\hat{U}_t(g(A,\mathbf{\Theta}))$ on $\overline{V}_t(\Lambda)$ by

$$\begin{aligned}
d\ (f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda) &= -\sum_{s=1}^{p} \delta_{i_s 0}\ f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda, \\
k_i\ (f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda) &= q^{(\Lambda - \sum_{s=1}^{p} \alpha_{i_s},\ \alpha_i)}\ f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda, \\
f_i\ (f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda) &= f_i f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda, \\
e_i\ (f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda) &= \sum_{s=1}^{p} \delta_{i i_s} (-1)^{[e_i] \sum_{k=1}^{s-1} [f_{i_k}]}\ f_{i_1} f_{i_2} ... \hat{f}_{i_s} ... f_{i_p} v_+^\Lambda \\
&\times \frac{q^{(\Lambda - \sum_{r=s+1}^{p} \alpha_{i_r},\ \alpha_{i_s})} - q^{-(\Lambda - \sum_{r=s+1}^{p} \alpha_{i_r},\ \alpha_{i_s})}}{q^{\epsilon_{i_s}} - q^{-\epsilon_{i_s}}}.
\end{aligned} \tag{2}$$

All the relations of (1) are clearly satisfied, except the Serre relations amongst the e_i's. Set $S_{ij} = (Ade_i)^{1-a_{ij}}(e_j)$, $i \neq j$, It is a consequence of the 'quadratic' relations that $[S_{ij},\ f_k\} = 0$. Thus for all $i \neq j$,

$$S_{ij}(f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda) \quad = \quad 0,$$

and $\overline{V}_t(\Lambda)$ indeed yields a $\hat{U}_t(g(A,\mathbf{\Theta}))$ module.

This module is in general not irreducible, but contains a maximal proper submodule $M(\Lambda)$ such that

$$V_t(\Lambda) \quad = \quad \overline{V}_t(\Lambda)/M(\Lambda),$$

yields an irreducible $U_q(g(A,\mathbf{\Theta}))$ module, which is called an irreducible highest weight module with highest weight Λ. The image of v_+^Λ under the canonical projection is the

maximal vector of $V_t(\Lambda)$. Standard arguments show that up to isomorphisms, $V_t(\Lambda)$ is uniquely determined by its highest weight.

Following the terminology of the representation theory of Lie algebras, we call a $\hat{U}_t(g(A, \mathbf{\Theta}))$ module V_t integrable if all e_i and f_i act on V_t by locally nilpotent endomorphisms, namely, for any $v \in V_t$, there exists a nonnegative integer $m_v < \infty$ such that

$$(e_i)^{m_v} v = \quad (f_i)^{m_v} v = 0, \quad \forall i \in I.$$

Consider the irreducible highest weight $\hat{U}_t(g(A, \mathbf{\Theta}))$ module $V_t(\Lambda)$ with highest weight Λ and maximal vector v_+^Λ. It is obviously true that the e_i always act on $V_t(\Lambda)$ by locally nilpotent endomorphisms. However, nilpotency of the f_i action imposes strong conditions on the highest weight.

For a fixed $i \notin \mathbf{\Theta}$, the elements e_i, f_i and h_i generate a $U_{q^{\epsilon_i}}(sl(2))$ subalgebra of $\hat{U}_t(g(A, \mathbf{\Theta}))$. In order for $(f_i)^m v_+^\Lambda$ to vanish, Λ must satisfy the condition $\frac{2(\Lambda,\ \alpha_i)}{(\alpha_i,\ \alpha_i)} \in \mathbf{Z}_+$. When $\mu \in \mathbf{\Theta}$, we have normalized $(\alpha_\mu,\ \alpha_\mu) = 1$. Now $e = e_\mu$, $f = f_\mu$, $h = h_\mu$, generate a $U_q(osp(1|2))$ subalgebra,

$$[h,\ e] = e, \quad [h,\ f] = -f, \quad ef + fe = \frac{q^h - q^{-h}}{q - q^{-1}}.$$

Since $f^m v_+^\Lambda = 0$ for a large enough m, but $v_+^\Lambda \neq 0$, there must exist an integer k, $0 < k < m$, such that $f^k v_+^\Lambda \neq 0$, and $f^{k+1} v_+^\Lambda = 0$. Applying e to $f^{k+1} v_+^\Lambda$, we arrive at

$$e f^{k+1} v_+^\Lambda \quad = \frac{q^{\frac{k+1}{2}} - q^{-\frac{k+1}{2}}}{(q - q^{-1})(q^{1/2} - q^{-1/2})} \left[q^{\frac{(\Lambda,\ \alpha_\mu)}{(\alpha_\mu,\ \alpha_\mu)} - \frac{k}{2}} - (-1)^k q^{-\frac{(\Lambda,\ \alpha_\mu)}{(\alpha_\mu,\ \alpha_\mu)} + \frac{k}{2}} \right] f^k v_+^\Lambda \quad = 0,$$

which requires $\frac{2(\Lambda,\ \alpha_\mu)}{(\alpha_\mu,\ \alpha_\mu)} = k$, and $k \in 2\mathbf{Z}_+$. In fact,
The irreducible highest weight $\hat{U}_t(g(A, \mathbf{\Theta}))$ module $V_t(\Lambda)$ with highest weight Λ is integrable if and only if

$$\begin{aligned} \frac{2(\Lambda,\ \alpha_i)}{(\alpha_i,\ \alpha_i)} &\in \mathbf{Z}_+, \quad \forall i \in I, \\ \frac{2(\Lambda,\ \alpha_\mu)}{(\alpha_\mu,\ \alpha_\mu)} &\in 2\mathbf{Z}_+, \quad \forall \mu \in \mathbf{\Theta}. \end{aligned} \tag{3}$$

Note the presence of the second condition requiring the Dynkin labels associated with the odd simple roots be non - negative *even* integers, which is not needed in

the case of ordinary quantum affine algebras. We prove the assertion following the strategy of [2]. As pointed out earlier, all the e_i act on $V_t(\Lambda)$ by locally nilpotent endomorphisms. We have also seen that under the given conditions of Λ, the maximal vector v_+^Λ of $V_t(\Lambda)$ is annihilated by a sufficiently high power of each f_i, $i \in I$. Now consider the element $w = f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda$. We use induction on p to prove the nilpotency of the action of the f_i on w. Assume that $x = f_{i_2} ... f_{i_p} v_+^\Lambda$ is annihilated by $(f_i)^m$, $\forall i \in I$. Then $(f_{i_1})^m w = (f_{i_1})^{m+1} x = 0$. For $j \neq i_1$, consider $(f_j)^{m-a_{ji_1}} w = (f_j)^{m-a_{ji_1}} f_{i_1} x$. By using the Serre relation $(Adf_j)^{1-a_{ji_1}}\ f_{i_1} = 0$, we can express $(f_j)^{m-a_{ji_1}} f_{i_1}$ as a $\mathbf{C}[[t]]$ linear combination of the elements $(f_j)^{-a_{ji_1}-\nu} f_{i_1} (f_j)^{m+\nu}$, $\nu = 0, 1, ..., -a_{ji_1}$, which all annihilate x. Hence, $(f_j)^{m-a_{ji_1}} w = 0$.

As $V_t(\Lambda)$ can be generated by repeatedly applying the f_i to the maximal vector v_+^Λ, a subset $\mathcal{B}$ of all the elements of the form $f_{i_1} f_{i_2} ... f_{i_p} v_+^\Lambda$ provides a basis of $V_t(\Lambda)$ over $\mathbf{C}[[t]]$. Hence we have proved that all e_i and f_i act on $V_t(\Lambda)$ by locally nilpotent endomorphisms, and $V_t(\Lambda)$ is integrable.

Let $V(\Lambda)$ be the vector space over $\mathbf{C}$ with the basis $\mathcal{B}$. Then as a $\mathbf{C}[[t]]$ module, $V_t(\Lambda) = V(\Lambda)[[t]]$. Our earlier discussions assert that $V(\Lambda) \cong V_t(\Lambda)/tV_t(\Lambda)$ carries a natural $U(g(A, \mathbf{\Theta}))$ module structure, and $V_t(\Lambda)$ is a deformation of $V(\Lambda)$. It follows the integrability of $V_t(\Lambda)$ that $V(\Lambda)$ is integrable as a $U(g(A, \mathbf{\Theta}))$ module. It is also of highest weight type, and is cyclically generated by a the maximal vector with weight Λ. A result of [1] states that an integrable $U(g(A, \mathbf{\Theta}))$ module is completely reducible. Thus $V(\Lambda)$, being cyclically generated, must be irreducible. Also recall that every integrable irreducible highest weight $U(g(A, \mathbf{\Theta}))$ module is uniquely determined by an element $\Lambda \in H^*$ satisfying the same conditions as (3). Thus,

Every irreducible integrable highest weight $\hat{U}_t(g(A, \mathbf{\Theta}))$ module is a deformation of an irreducible integrable highest weight $U(g(A, \mathbf{\Theta}))$ module with the same highest weight, and all such irreducible $U(g(A, \mathbf{\Theta}))$ modules can be deformed.

Note that the integrable lowest weight irreps of the quantum affine superalgebras can be studied in the same way, and the above result applies as well. It should also be mentioned that Kac' character formula [1] for the integrable highest weight irreps of $U(g(A, \mathbf{\Theta}))$ still works in the quantum case.

IV. Jimbo Model and Bose - Fermi Transmutation

The Jimbo version of quantum affine superalgebra $U_q(g(A,\mathbf{\Theta}))$ is a $\mathbf{Z}_2$ graded associative algebra over the complex number field $\mathbf{C}$, generated by the elements $\{d,\ k_i,\ k_i^{-1},\ e_i,\ f_i,\ i\in I\}$, subject to the same relations as (1), but with q now being regarded as a non - zero complex parameter. Nevertheless, it is possible to formulate the Jimbo type 'quantization' within the framework of deformation theory [5].

We still set $q = exp(t)$, and let $t_i = t\epsilon_i$. Define

$$\begin{aligned} S_i &= \frac{k_i - k_i^{-1}}{q^{\epsilon_i} - q^{-\epsilon_i}}, \\ C_i &= \frac{k_i + k_i^{-1}}{2}. \end{aligned}$$

The relations of (1) involving $k_i^{\pm 1}$ can now be re - expressed in terms of S_i and C_i,

$$\begin{aligned} [d, C_i] = 0, &\qquad [d, S_i] = 0, \\ C_i S_j = S_j C_i, &\qquad (C_i)^2 - (S_i)^2\ sinh^2 t_i = 1, \\ C_i e_j - e_j C_i\ cosh[t(\alpha_i,\ \alpha_j)] &= e_j S_i\ sinh t_i\ sinh[t(\alpha_i,\ \alpha_j)], \\ S_i e_j - e_j S_i\ cosh[t(\alpha_i,\ \alpha_j)] &= e_j C_i\ sinh[t(\alpha_i,\ \alpha_j)]/sinh t_i, \\ C_i f_j - f_j C_i\ cosh[t(\alpha_i,\ \alpha_j)] &= -f_j S_i\ sinh t_i\ sinh[t(\alpha_i,\ \alpha_j)], \\ S_i f_j - f_j S_i\ cosh[t(\alpha_i,\ \alpha_j)] &= -f_j C_i\ sinh[t(\alpha_i,\ \alpha_j)]/sinh t_i, \\ [e_i,\ f_j\} &= \delta_{ij} S_i, \end{aligned}$$

while the Serre relations remain the same. We can now regard $U_q(g(A,\mathbf{\Theta}))$ as generated by $d,\ C_i,\ S_i,\ e_i,\ f_i$ for any $t\in\mathbf{C}$. Furthermore, for a fixed $t_0\in\mathbf{C}$, and $t = t_0+\tau$, we can consider τ as a formal parameter, and define the formal Jimbo quantum affine superalgebra $U_q(g(A,\mathbf{\Theta}))$ as a properly completed $\mathbf{C}[[\tau]]$ algebra generated by $d,\ C_i,\ S_i,\ e_i,\ f_i$ with the same relations. Then $U_q(g(A,\mathbf{\Theta}))$ is a deformation of $U_{exp(t_0)}(g(A,\mathbf{\Theta}))$.

At $t_0 = 0$, $U_{exp(t_0)}(g(A,\mathbf{\Theta}))$ is isomorphic to an extension of the universal enveloping algebra of $g(A,\mathbf{\Theta})$ by the C_i, which satisfy $C_i^2 = 1$. Explicitly,

$$U_1(g(A,\mathbf{\Theta})) \ = \ U(g(A,\mathbf{\Theta}))\otimes\mathbf{CZ_2^{\otimes(n+1)}},$$

where $\mathbf{CZ}_2^{\otimes(n+1)}$ is the group algebra of the abelian group generated by the C_i.

Therefore, strictly speaking, the Jimbo model of $U_q(g(A, \mathbf{\Theta}))$ is not a deformation of $U(g(A, \mathbf{\Theta}))$, but rather an an extension of $U(g(A, \mathbf{\Theta}))$ by some parity operators.

More interesting is the case when $t_0 = i\pi$. At $\tau = 0$, the relations become

$$\begin{aligned}
[d, C_i] = 0, &\qquad [d, S_i] = 0, \\
C_i S_j = S_j C_i, &\qquad (C_i)^2 = 1, \\
C_i e_j = (-1)^{(\alpha_i,\ \alpha_j)}\ e_j C_i, &\qquad C_i f_j = (-1)^{(\alpha_i,\ \alpha_j)}\ f_j C_i, \\
S_i e_j - (-1)^{(\alpha_i,\ \alpha_j)}\ e_j S_i &= (-1)^{(\alpha_i,\ \alpha_j)+\epsilon_i}\ (\alpha_i,\ \alpha_j)\ e_j C_i, \\
S_i f_j - (-1)^{(\alpha_i,\ \alpha_j)}\ f_j S_i &= -(-1)^{(\alpha_i,\ \alpha_j)+\epsilon_i}\ (\alpha_i,\ \alpha_j)\ f_j C_i, \\
[e_i,\ f_j\} &= \delta_{ij} S_i.
\end{aligned}$$

and the Serre relations read

$$(ade_i)^{1-a_{ij}}(e_j) = 0, \quad (adf_i)^{1-a_{ij}}(f_j) = 0, \quad i \neq j,$$

with

$$\begin{aligned}
ade_i(x) &= e_i x - (-1)^{(\alpha_i,\ \omega(x))+[x][e_i]} x e_i, \\
adf_i(x) &= f_i x - (-1)^{(\alpha_i,\ \omega(x))+[x][e_i]} x f_i.
\end{aligned}$$

These are the defining relations for the complex associative algebra $U_{-1}(g(A, \mathbf{\Theta}))$, which, unfortunately, are rather complicated, and not very illuminating. However, by applying certain inner automorphisms constructed out of the C_i, we can cast the relations into a more familiar form.

For definiteness, let us consider $B^{(1)}(0, n)$. Set $\sigma_i = \prod_{k=i}^{n} C_k$, and define

$$\begin{aligned}
D &= d, \\
H_i &= (-1)^{\epsilon_i} C_i\ S_i, \\
E_i &= \sigma_{i+1}\ (\sigma_1)^{\delta_{i0}}\ e_i, \\
F_i &= \sigma_i\ (\sigma_1)^{\delta_{i0}}\ f_i.
\end{aligned}$$

Now something rather intriguing happens: these elements do *not* obey the defining relations of the affine superalgebra $B^{(1)}(0, n)$, instead they generate the universal enveloping algebra of the twisted ordinary (i.e., non - graded) affine Lie algebra $A_{2n}^{(2)}$.

Recall that the universal enveloping algebra of $A_{2n}^{(2)}$ and that of $B^{(1)}(0,n)$ are totally different algebraic structures, although their underlying vector spaces (ignoring the $\mathbf{Z}_2$ grading in the case of $B^{(1)}(0,n)$) are isomorphic. Nevertheless, $U_q(B^{(1)}(0,n))$ can be obtained as a deformation of the universal enveloping algebra of $A_{2n}^{(2)}$ supplemented by $n+1$ parity operators: a kind of transmutation between the ordinary affine algebra (which is boson - like) and affine superalgebra (which is fermion - like) takes places upon quantization. Such a transmutation was found in the case of $osp(1|2n)$ and $so(2n+1)$ in [7], providing a natural explanation for the observation made by Rittenberg and Scheunert [8] that there was a one to one correspondence between the tensorial irreducible representations of $so(2n+1)$ and the finite dimensional irreducible representations of $osp(1|2n)$.

Note that the C_i generate the group algebra of the abelian group $\mathbf{Z}_2^{\otimes(n+1)}$. When acting by conjugation on the elements of the $A_{2n}^{(2)}$ generators, they give rise to parity factors, i.e., $\pm$ signs. We introduce the notation $U(A_{2n}^{(2)}) \bowtie \mathbf{CZ}_2^{\otimes(n+1)}$ to illustrate the fact that $U_{-1}(B^{(1)}(0,n))$ is the universal enveloping algebra of $A_{2n}^{(2)}$ supplemented by the C_i.

A case by case study shows that such Bose - Fermi transmutation occurs with other affine superalgebras as well; we have

$$\begin{aligned}
U_{-1}(B^{(1)}(0,n)) &\cong U(A_{2n}^{(2)}) \bowtie \mathbf{CZ}_2^{\otimes(n+1)}, \quad n>1,\\
U_{-1}(B^{(1)}(0,1)) &\cong U(A_{2}^{(2)}) \bowtie \mathbf{CZ}_2^{\otimes 2},\\
U_{-1}(A^{(2)}(0,2n-1)) &\cong U(B_{n}^{(1)}) \bowtie \mathbf{CZ}_2^{\otimes(n+1)}, \quad n>2,\\
U_{-1}(A^{(2)}(0,3)) &\cong U(C_{2}^{(1)}) \bowtie \mathbf{CZ}_2^{\otimes 3},\\
U_{-1}(C^{(2)}(n+1)) &\cong U(D_{n+1}^{(2)}) \bowtie \mathbf{CZ}_2^{\otimes(n+1)},\\
U_{-1}(C^{(2)}(2)) &\cong U(A_{1}^{(1)}) \bowtie \mathbf{CZ}_2^{\otimes 2}.
\end{aligned} \tag{4}$$

However, the $A^{(4)}(0,2n)$ series proves to be an exception

$$U_{-1}(A^{(4)}(0,2n)) \cong U(A^{(4)}(0,2n)) \bowtie \mathbf{CZ}_2^{\otimes(n+1)}, \quad n=1,2,..., \tag{5}$$

where no Bose - Fermi transmutation has been observed.

The transmutation between ordinary quantum affine algebras and quantum affine superalgebras can also be realized at the level of representations. Consider an irre-

ducible integrable highest weight module $V_t(\Lambda)$ of the Drinfeld quantum affine superalgebra $\hat{U}_t(g(A, \mathbf{\Theta}))$ studied in the last section. Note that t enters the formulae (2) through q, thus by specializing t to a complex number $t = i\pi + \tau$, with τ a generic complex parameter, we obtain from $V_t(\Lambda)$ a module of the Jimbo quantum affine superalgebra $U_q(g(A, \mathbf{\Theta}))$, which we denote by $\check{V}_q(\Lambda)$. If $g(A, \mathbf{\Theta})$ is one of the affine superalgebras appearing in (4), then the representation of $U_q(g(A, \mathbf{\Theta}))$ furnished by $\check{V}_q(\Lambda)$ can be realized by an irreducible integrable highest weight representation of the ordinary quantum affine algebra $U_{-q}(g(A, \emptyset))$, where $g(A, \emptyset)$, appearing on the right hand sides of (4), is the ordinary affine Lie algebra with the same Cartan matrix A, but with all generators being even.

For the sake of concreteness, consider again the case of $U_q(B^{(1)}(0, 2n))$. Denote by $D,\ E_i,\ F_i,\ (K_i)^{\pm 1}$ the generators of $U_{-q}(A^{(2)}_{2n})$, while the generators of $U_q(B^{(1)}(0, 2n))$ are still denoted by $d,\ e_i,\ f_i,\ (k_i)^{\pm 1}$. Since the Cartan subalgebras of $B^{(1)}(0, 2n)$ and $A^{(2)}_{2n}$ are isomorphic, we will make no distinctions between them.

Let $\check{W}_{-q}(\Lambda)$ be an irreducible $U_{-q}(A^{(2)}_{2n})$ module with highest weight Λ satisfying the conditions (3). As a complex vector space $\check{W}_{-q}(\Lambda)$ admits the weight space decomposition

$$\check{W}_{-q}(\Lambda) \ = \ \bigoplus_{\omega \leq \Lambda} W^\omega,$$

where each W^ω is finite dimensional, and $(\alpha_i,\ \omega) \in \mathbf{Z}$, $\forall i \in I$. Define a $U_q(B^{(1)}(0, 2n))$ action on $\check{W}_{-q}(\Lambda)$ by

$$\begin{aligned}
dw &= Dw, \\
k_i w &= (-1)^{(\alpha_i,\ \omega)} K_i w, \\
e_i w &= (-1)^{(\beta_{i+1} - \beta_1 \delta_{i0},\ \omega + \alpha_i)} E_i w, \\
f_i w &= (-1)^{(\beta_i - \beta_1 \delta_{i0},\ \omega + \alpha_i)} F_i w, \qquad \forall w \in W^\omega,
\end{aligned}$$

where $\beta_i = \sum_{r=i}^n \alpha_r$. Direct calculations show that this definition indeed preserves the defining relations of $U_q(B^{(1)}(0, 2n))$, thus turning $\check{W}_{-q}(\Lambda)$ into a $U_q(B^{(1)}(0, 2n))$ module. This module is clearly irreducible, and has highest weight Λ. Thus it is isomorphic to $\check{V}_q(\Lambda)$. Observe that the subset of H^* satisfying (3) exhausts all the integral dominant weights for $B^{(1)}(0, 2n)$. Therefore, every irreducible integrable highest weight representation of $U_q(B^{(1)}(0, 2n))$ can be realized this way.

In a similar way we can show that the same result also holds for other affine superalgebras:

Let $g(A, \mathbf{\Theta})$ be an affine superalgebra appearing in (4). Then each irreducible integrable highest weight representation of $U_q(g(A, \mathbf{\Theta}))$ can be realized by a representation of $U_{-q}(g(A, \ \emptyset))$ of the same kind.

However, the converse is not true. There exist integrable irreps of $U_{-q}(g(A, \ \emptyset))$ with highest weights not satisfying the second condition of (3).

References

[1] V. G. Kac, Adv. Math. **30** (1978) 85.

[2] G. Lusztig, Adv. Math. **70** (1988) 237.

[3] M. Gerstenhaber, Ann. Math. **78** (1963) 267.

[4] V. G. Drinfeld, Leningrad Math. J. **1** (1989) 321.

[5] P. Bonneau, M. Flato, M. Gerstenhaber and G. Pinczon, Commun. Math. Phys., **161** (1994) 125.

[6] M. Lesimple and G. Pinczon, J. Math. Phys., **34** (1993) 4251.

[7] R. B. Zhang, Lett. Math. Phys., **25** (1992) 317.

[8] V. Rittenberg and M. Scheunert, Commun. Math. Phys., **83** (1982) 1.